Journal of Applied Mechanics and Technical Physics, Vol. 40, No. 4, 1999

FLOWS WITH BOUNDARY LAYERS IN UNBOUNDED REGIONS

V. V. Kuznetsov

UDC 532.526/532.61

Flows with Marangoni boundary layers having differently directed velocities of the external flow and the tangent stress at the free boundary are considered. The conditions of occurrence of counterflows are studied. An analog of the system of Prandtl equations near the contact point of three phases is obtained. Examples of the solutions are given.

Introduction. It is known that one can separate Prandtl boundary layers in the vicinity of the rigid walls and Marangoni layers near the free boundaries when a sufficiently intense motion of the fluid is described. The problem of the Marangoni boundary layer was formulated by Napolitano [1] and was studied from various viewpoints [2–4], mainly in connection with applications to the problems of space materials science. In particular, Napolitano and Golia [2] studied invariant solutions of this problem. However, only the case was considered where the tangent stress at a free surface, which induces the boundary layer, is directed in the direction of the fluid flow. Since this stress frequently causes the motion of the fluid, this case can be regarded as a basic case. Nevertheless, the tangent stress can be opposite in direction to the main flow. After that, a counterflow can occur in the motion region.

In the present study, we investigate the conditions of occurrence of a counterflow in Marangoni boundary layers when the tangent stresses are oppositely directed to the main fluid flow and in the layers conjugate to the main layer whose external flow is a quiescent state. The solutions of the Cauchy problem for the Blasius equation are refined. An approximate formula to determine the distance from the coordinate origin to the point of onset of a counterflow for specified constant and differently directed velocities of the main layer and the tangent stress at the free boundary is derived. An analog of the system of Prandtl equations for the description of flows with large Reynolds numbers near the three-phase contact point is obtained. For this system, the problem is formulated, and examples of the solutions are given.

1. Formulation of the Problem. For a Marangoni flat stationary boundary layer, the basic problem has the following form: in the domain $x > 0$, $y > 0$, it is necessary to determine the velocity components u and v that satisfy the system

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\varrho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1.1}$$

with the boundary conditions

$$u\Big|_{x=0} = u_0(y), \quad \varrho\nu\frac{\partial u}{\partial y}\Big|_{y=0} = f(x), \quad v\Big|_{y=0} = 0, \quad u(x,y) \to U(x) \quad \text{as} \quad y \to \infty. \tag{1.2}$$

In (1.1) and (1.2), the density is $\varrho = \text{const}$, the functions u_0, f, and U are the parameters of the problem, and the pressure $p(x)$ is connected with the velocity $U(x)$ of the external flows by the relation (the Bernoulli integral) $2p(x)/\varrho + U^2(x) = \text{const}$; the function $f(x)$ has the meaning of the tangent stress along the free surface, which can be caused, in particular, by the thermocapillary effect. Hereinafter, we consider that $U(x) \geqslant 0$. The tangent stress on the free surface is directed streamwise if $f(x) \leqslant 0$ and counterstream if $f(x) > 0$. Generally speaking, in this case, one can expect the presence of a counterflow zone. The questions

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 40, No. 4, pp. 69–80, July–August, 1999. Original article submitted May 29, 1997; revision submitted October 29, 1997.

0021-8944/99/4004-0611 $22.00

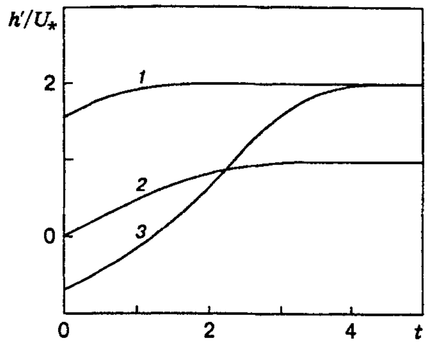

Fig. 1

arise concerning the relation between the parameters of the problem at which a counterflow can occur, the part of the boundary layer involved into this counterflow, and the thickness of the entire Marangoni boundary layer.

In the theory of Prandtl boundary layer, to estimate the thickness of the boundary layer, the so-called displacement thickness [5] is generally used. It is not applicable to Marangoni boundary layers, because, in this case, $U = 0$ for some values of x. We shall introduce the analog of this quantity, namely, the *perturbation thickness* δ^*, by means of the equality

$$[u(x,0) - U(x)]\delta^* = \int_0^\infty [u(x,y) - U(x)]\, dy. \tag{1.3}$$

The introduction of the new term is justified not only owing to a certain difference in the definition but also in the physical meaning: the Marangoni boundary layer is the velocity perturbation near the free boundary; at the same time, it is difficult to imagine that, just as a solid, the free boundary "displaces" the streamlines.

2. Interaction between the Uniform Flow and the Surface Stress. Let $U = \text{const}$ and $f(x) = F/\sqrt{x}$. We search for a solution of problem (1.1), (1.2) in the form $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. The stream function is $\psi = (F\nu^2)^{1/3}\sqrt{x}h(t)$, where $t = (F/\nu)^{1/3}y/\sqrt{x}$. In this case, the function h satisfies the Blasius equation [5]

$$2h''' + hh'' = 0 \tag{2.1}$$

with the boundary conditions

$$h(0) = 0, \qquad h''(0) = 1, \qquad h' \to \frac{U}{(F^2\nu)^{1/3}} \quad \text{as} \quad t \to \infty. \tag{2.2}$$

The calculations show that, if $U = U_* \approx 2.085(F^2\nu)^{1/3}$, there is a solution h_* of problem (2.1), (2.2) such that $h'_*(0) = 0$, i.e., the free surface is fixed, and this solution differs from the solution of the classical Blasius problem [5] only by the continuation of the variables.

For each $U > U_*$, one can construct two solutions of problem (2.1), (2.2); $h'(t) > 0$ in one of them for all t, and the other solution has a domain of change of the argument where $h'(t) < 0$. The examples of calculation of problem (2.1), (2.2) are given in Fig. 1 in the form of graphs of the function $h'(t)$ for certain values of U. Curve 2 corresponds to $U = U_*$, and curves 1 and 3 to $U = 2U_*$. Since the equilibrium at the free boundary is reached for $U = U_*$ for a given f, one can assume that the longitudinal velocity is positive everywhere for $U > U_*$; therefore, we consider that the flows which correspond to curve 3 cannot occur.

To find out what will happen for $U < U_*$, it is necessary to refine the properties of the solutions of the Cauchy problem for the Blasius equation (2.1) with the initial data

$$h(0) = 0, \qquad h'(0) = \beta, \qquad h''(0) = \gamma. \tag{2.3}$$

It is known [6, Chapter 14] that problem (2.1), (2.3) for $\beta \geqslant 0$ is uniquely solved; note that $h'' > 0$

(if $\gamma > 0$) and there exists a limit $h'(t)$ as $t \to \infty$, which is denoted in terms of $U(\beta,\gamma)$. Here $U(\beta,\gamma)$ is a continuous function of argument γ, and, if $\gamma_1 > \gamma_2$, we have $U(\beta,\gamma_1) > U(\beta,\gamma_2)$. We shall show that problem (2.1), (2.3) can be solved for negative β.

Lemma 1. *Problem (2.1), (2.3) can be solved "as a whole" for $\beta < 0$ and $\gamma > 0$ and there exist $t^* > 0$ and $t^0 \in (0,t^*)$ such that $h(t) < 0$ for $t \in (0,t^*)$ and $h(t) > 0$ for $t > t^*$, $h'(t) < 0$ for $t \in (0,t^0)$, and $h'(t) > 0$ for $t > t^0$.*

Proof. The solution of the Cauchy problem (2.1), (2.3) can be considered as a solution of the system

$$h' = w, \qquad w' = a, \qquad a' = -ha/2$$

which passes through the point $t = 0$, $h = 0$, $w = \beta$, and $a = \gamma$. Since the right-hand sides of the equations are continuous Lipshits functions of their arguments, the problem can be solved locally and this solution can be continued to the domain $t > 0$ or over the entire numerical axis, or until the right-hand sides are restricted.

It follows from Eq (2.1) [this equation is linear relative to $h''(t)$] that

$$h''(t) = \gamma \exp\Big\{-\frac{1}{2}\int_0^t h(\tau)d\tau\Big\}.$$

In addition,

$$h'(t) = \beta + \int_0^t h''(\tau)d\tau.$$

Then h' and h'' can become infinite for a certain value of the argument $t = t'$ only together with $h(t')$. Therefore, the unlimitedness of the right-hand sides of the system means the presence of a vertical asymptote in the graph of the function $h(t)$. However, $t^0 \in (0,t^*)$ and $h'(t^0) = 0$ exist, and since the graph $h(t)$ is convex downward, $h \underset{t\to t'}{\longrightarrow} +\infty$, there is a value of $t = t^*$ such that $h(t^*) = 0$ and $h(t)$ is smooth. In addition, if one denotes $\beta^* = h'(t^*)$ and $\gamma^* = h''(t^*)$, this solution is invariant under the shift of the reference point of the argument, since the argument does not enter explicitly into Eq. (2.1). As a result, the solution will be also the solution of the Cauchy problem (2.1), (2.3), which corresponds to the $\beta = \beta^* > 0$ and $\gamma = \gamma^* > 0$; therefore, it can be continued infinitely to the right.

This contradiction shows that the solution $h(t)$ can be continued to the entire numerical semi-axis. The existence of the finite limit $U(\beta,\gamma)$ of the function $h'(t)$ as $t \to \infty$ can be proved as done in [6] for positive β. It is clear that, if $U(\beta,\gamma) > 0$, we again obtain the existence of t^* and t^0 such that $h(t^*) = 0$ and $h'(t^0) = 0$. Since $h'(t)$ increases monotonically, the conditions that the solution and its derivative are sign-determined, which are required by Lemma 1, are satisfied, and Lemma 1 can be considered proved in this case.

If $U(\beta,\gamma) \leqslant 0$, we have $h'''(t) = -0.5h(t)h''(t) > 0$ everywhere, because the maxima $h(t) < 0 \;\forall\, t > 0$ are impossible. Since $h''(0) = \gamma > 0$, we have $h''(t) > \gamma$ and $U(\beta,\gamma) = \lim\limits_{t\to\infty} h'(t)$ cannot exist. Therefore, $U(\beta,\gamma)$ is positive. Lemma 1 is proved.

We now establish the properties of the monotonicity of $U(\beta,\gamma)$ relative to the first argument.

Lemma 2. *If $\beta_1 > \beta_2 \geqslant 0$ or $\beta_1 < \beta_2 \leqslant 0$, then we have $U(\beta_1,\gamma) \geqslant U(\beta_2,\gamma)$.*

Proof. We shall pass to the new independent variable $w = h'(t)$ in problem (2.1), (2.3). We obtain

$$w = \frac{dh}{dt} = \frac{dh}{dw}\frac{dw}{dt} = \dot{h}(w)h''(t), \qquad h''(t) = \frac{w}{\dot{h}}, \qquad \dot{h}(w) = \frac{w}{h''(t)}. \tag{2.4}$$

Hereinafter, the dot and the prime denote differentiation with respect to w and t, respectively. For the function $h(w)$, we obtain the Cauchy problem

$$w\ddot{h} = \dot{h} + \frac{1}{2}h\dot{h}^2, \tag{2.5}$$

$$h(\beta) = 0, \qquad \dot{h}(\beta) = \beta/\gamma. \tag{2.6}$$

By virtue of the properties of the function $h(t)$, the map $w\colon [0,\infty) \longrightarrow [\beta, U(\beta,\gamma))$ is homeomorphism; therefore, there is a one-to-one correspondence between the solutions of problems (2.1), (2.3) and (2.5), (2.6). As a result, it follows from the properties of the function $h(t)$ that $\dot{h}(w) < 0$ for $w < 0$ and $\dot{h}(w) > 0$ for $w > 0$, and the graph $h(w)$ is convex downward and intersects the ordinate at a right angle. The fact that $h'(t) \to U(\beta,\gamma)$ as $t \to \infty$ means that $h(w)$ has a vertical asymptote as $w \to U(\beta,\gamma)$.

Let $\beta_1 > \beta_2 \geqslant 0$. Then, if $h_1(t), h_2(t)$ and $h_1(w), h_2(w)$ are the corresponding solutions of problems (2.1), (2.3) and (2.5), (2.6), we have $h_2(\beta_1) > h_1(\beta_1) = 0$, because $\dot{h}_2(w) > 0$. In addition, according to (2.4), we have $\dot{h}_2(\beta_1) = \beta_1/h_2''(t(\beta_1))$, where $t(w)$ denotes the value of t that corresponds to the given $w = h_2'(t)$. However, $h_2'''(t) = -0.5h_2(t)h_2''(t) < 0$ and, therefore, we have $h_2''(t) < \gamma$. As a result, we have $\dot{h}_2(\beta_1) > \beta_1/\gamma = \dot{h}_1(\beta_1)$. Equation (2.5) is equivalent to the system

$$\dot{h} = W, \qquad \dot{W} = \frac{W + hW^2/2}{w}. \tag{2.7}$$

It is obvious that the right sides of system (2.7) are increasing functions of the arguments h and W for $h > 0$ and $w > 0$. As is shown above, $h_2(\beta_1) > h_1(\beta_1)$ and $W_2(\beta_1) > W_1(\beta_1)$. Therefore, we have $h_2(w) > h_1(w)$ everywhere for $w \geqslant \beta_1$; moreover, the difference $h_2 - h_1$ increases [6, Chap. 3]. This means that the graph of the function $h_1(w)$ lies below that of the function $h_2(w)$ and, consequently, has a vertical asymptote farther to the left of the graph h_2, i.e., $U(\beta_1,\gamma) \geqslant U(\beta_2,\gamma)$, as was to be shown.

If $\beta_1 < \beta_2 \leqslant 0$, according to the properties of the solution $h(w)$, each solution of problem (2.5), (2.6) is also the solution of the Cauchy problem for Eq. (2.5) with the initial data

$$h(0) = h^0 < 0, \qquad \dot{h}(0) = 0. \tag{2.8}$$

Therefore, $h_1(w)$ and $h_2(w)$ are also the solutions of problem (2.5), (2.8) with certain values h_1^0 and h_2^0 of the parameter h^0. We shall show that $h_1^0 \leqslant h_2^0$. We have $\dot{h}_1(\beta_2) = \beta_2/h_1''(t(\beta_2)) > \beta_2/\gamma = \dot{h}_2(\beta_2)$, because $h_1''(0) = \gamma$, and $h_1'''(t) = -0.5h_1(t)h_1''(t) > 0$ for t, where $h_1(t) < 0$. Since h_1 and h_2 are the solutions of Eq. (2.5), their difference $H = h_2 - h_1$ satisfies the equation

$$w\ddot{H} - \dot{H}\Big[1 + \frac{1}{2}h_1(\dot{h}_1 + \dot{h}_2)\Big] = -\frac{1}{2}\dot{h}_1^2 H, \tag{2.9}$$

which is linear relative to $\dot{H}$. Therefore, for $w \in (\beta_2, 0)$, we obtain

$$\dot{H}(w) = \dot{H}(\beta_2) - \frac{1}{2}e^{\zeta(w)} \int\limits_{\beta_2}^{w} e^{-\zeta(\tau)} H(\tau)\dot{h}_1^2(\tau)/\tau\, d\tau, \qquad \zeta(w) = \int\limits_{\beta_2}^{w} \Big[1 + \frac{1}{2}h_1(\dot{h}_1 + \dot{h}_2)\Big]\Big/\tau\, d\tau.$$

If there was a point $\beta_* \in (\beta_2, 0)$ of intersection of the graphs of h_1 and h_2, there would be $H > 0$ for $w < \beta_*$ and $H \leqslant 0$ for $w > \beta_*$. Obviously, in this case, $\dot{H}(\beta_*) \leqslant 0$ and the equality

$$\dot{H}(w) = \dot{H}(\beta_2) - \frac{1}{2}e^{\zeta(w)} \int\limits_{\beta_2}^{\beta_*} e^{-\zeta(\tau)} H(\tau)\dot{h}_1^2(\tau)/\tau\, d\tau - \frac{1}{2}e^{\zeta(w)} \int\limits_{\beta_*}^{w} e^{-\zeta(\tau)} H(\tau)\dot{h}_1^2(\tau)/\tau\, d\tau,$$

on the right side of which the first term is negative, the second is positive (their sum is not greater than zero), and the third is negative, would hold. Therefore, there would be $\dot{H}(0) = \dot{h}_2(0) - \dot{h}_1(0) < 0$, which contradicts (2.8).

As a result, we obtain $h_1^0 \leqslant h_2^0$. If $h_1^0 < h_2^0$, we have

$$\dot{H}(w) = \frac{1}{2}e^{\zeta(w)} \int\limits_{0}^{w} e^{-\zeta(\tau)} H(\tau)\dot{h}_1^2(\tau)/\tau\, d\tau > 0$$

for $w > 0$, i.e., the difference $h_2 - h_1$ does not decrease, and the graph $h_1(w)$ lies below than the graph $h_2(w)$, and, hence, $U(\beta_1,\gamma) \geqslant U(\beta_2,\gamma)$. If $h_1^0 = h_2^0$, we have $H(0) = \dot{H}(0) = 0$. It follows from (2.4) that $h(w) = O(w)$ as $w \to 0$; therefore, considering (2.9) for $w \nearrow 0$, we obtain $\ddot{H}(0) > 0$ with allowance for the fact that there

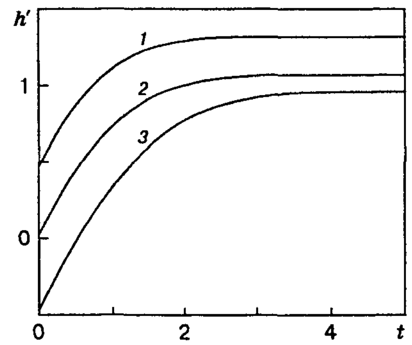

Fig. 2

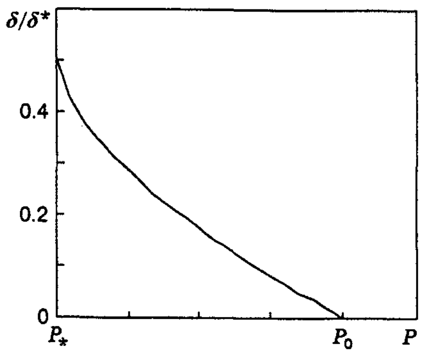

Fig. 3

should be $\dot{H} < 0$ as $w < 0$. Consequently, similarly to the above case, the difference $h_2 - h_1$ does not decrease for positive w and the statement of Lemma 2 holds. Lemma 2 is proved.

Corollary. *If $U < U_*$, smooth bounded solutions of problem* (2.1), (2.2) *do not exist.*

Proof. We assume that there is a solution $h(t)$ of problem (2.1), (2.2). Three variants are possible: $h'(0) > 0$, $h'(0) = 0$, and $h'(0) < 0$. According to Lemma 2, $U \geqslant U_*$ in all the cases, which contradicts the conditions. This contradiction proves the corollary.

Remark. It follows from the aforesaid that, for $U > U_*$, the two classes of solutions of problem (2.1), (2.2) (curves 1 and 3 in Fig. 1) are not different: in essence, they are the same solutions of Eq. (2.1) but shifted along the coordinate t.

We calculate the perturbation thickness for the Marangoni layer. We obtain

$$\delta^* = \frac{(\nu F)^{1/3}\sqrt{x}}{h'(0) - U/(F^2\nu)^{1/3}} \int_0^\infty (h' - U/(F^2\nu)^{1/3})\, dt.$$

3. Interaction between the Surface Stress and the Pressure Gradient. Let the pressure fall off downstream with a constant gradient P, i.e., $p = -Px$ and $U(x) = \sqrt{2Px/\varrho}$. Then, for $f(x) = Fx^{1/4}$, one can search for the stream function in the form

$$\psi(x,y) = \left(\frac{F\nu}{\varrho}\right)^{1/4} x^{3/4} h(t), \qquad t = \frac{y}{x^{1/4}}\left(\frac{F}{\varrho\nu^2}\right)^{1/3}.$$

To determine $h(t)$, we obtain the boundary-value problem

$$\frac{1}{2}h'^2 - \frac{3}{4}hh'' = P\left(\frac{\varrho\nu}{F^4}\right)^{1/3} + h'''; \tag{3.1}$$

$$h(0) = 0, \qquad h''(0) = 1, \qquad h' \to \sqrt{2P}(\varrho\nu/F^4)^{1/6} \quad \text{as} \quad t \to \infty. \tag{3.2}$$

The calculations show that there is a critical value of $P_* \approx 0.465(\varrho\nu)^{-1/3}F^{4/3}$ such that, if $P < P_*$, problem (3.1), (3.2) has no solution. However, in contrast to the results given in Sec. 2, the solution h_* of this problem, which corresponds to $P = P_*$, does not correspond to the case where the longitudinal velocity on the free surface vanishes [$h'(0) = 0$]. The value of $P = P_0 \approx 0.576(\varrho\nu)^{-1/3}F^{4/3}$ corresponds to this case; note that, if $P > P_0$, we have $h'(t) > 0$ for all $t \geqslant 0$, and, if $P_* \leqslant P < P_0$, we have $h'(t) < 0$ on a certain interval $(0, \delta)$ of t variation and $h'(t) > 0$ for $t > \delta$. This means that there is a counterflow zone in which the liquid flows toward the main stream near the free surface.

Examples of the calculation of problem (3.1), (3.2) are given in Fig. 2 where the graphs of the function $h'(t)$ for certain values of the parameter P are depicted. Curve 3 corresponds to $P = P_*$, curve 2 to $P = P_0$, and curve to $P = 3P_0/2$.

Figure 3 shows the thickness δ of the counterflow layer measured in fractions of the perturbation thickness δ^* versus P for $P \in [P_*, P_0]$. One can see that the counterflow can involve a significant part of the boundary layer. In this case, we have

$$\delta^* = \frac{(\varrho\nu^2/F)^{1/3}x^{1/4}}{h'(0) - \sqrt{2P(\varrho\nu/F^4)^{1/3}}} \int\limits_0^\infty [h' - \sqrt{2P(\varrho\nu/F^4)^{1/3}}]\, dt.$$

If the pressure gradient is not constant and is a certain power of the coordinate x, one can construct examples of the solutions with counterflows. These cases are omitted in this paper.

4. Onset of the Counterflow. It follows from the results of Sec. 2 that, if $U = \text{const}$, it is necessary that the tangent stress that initiates the boundary layer decrease inversely to the square root of the distance downstream to reach the equilibrium situation when $u = 0$ for $y = 0$. Clearly, if $f(x) = F = \text{const}$, a counterflow arises. Let us establish how far from the coordinate origin this will occur if the initial velocity profile $u_0(y)$ differs little from the constant U.

In this case, it is more convenient to solve problem (1.1)–(1.3) in dimensionless variables. Using U as the scale of velocity and $l_* = \nu/U$ as the scale of length during nondimensionalization and the same notation, we obtain the system

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4.1}$$

with the boundary conditions

$$u\Big|_{x=0} = u_0(y), \qquad \frac{\partial u}{\partial y}\Big|_{y=0} = \alpha, \qquad v\Big|_{y=0} = 0, \qquad u(x,y) \to 1 \quad \text{as} \quad y \to \infty. \tag{4.2}$$

Here $\alpha = F/(\varrho U^2)$. We use the function $u_0(y) = 1 - \mathrm{e}^{-k\alpha y}/k$ as the velocity profile. If k is large, u_0 differs slightly from unity and conditions (4.2) are consistent at the point $(0,0)$.

Problem (4.1), (4.2) was solved numerically by the grid method. It is noteworthy that the calculations in the Marangoni boundary layer for $u > 0$ are much simpler compared to those in the Prandtl layer, at least before the onset of a counterflow. This is due to the fact that here there are no difficulties associated with boundary-layer calculations, which were mentioned, e.g., in [7]. At the same time, the calculation cannot be continued indefinitely after the occurrence of a counterflow, because a situation similar to the "return time" for the heat-conduction equation arises. Since our aim is to find the distance from the coordinate origin to the counterflow zone, the latter circumstance is not important.

In the calculations, we employed the scheme

$$\frac{u_i^{j+1} - u_i^j}{\tau} + \frac{v_{i+1}^{j+1} - v_i^{j+1}}{h} = 0,$$

$$u_i^j \frac{u_i^{j+1} - u_i^j}{\tau} + v_i^j \frac{u_{i+1}^{j+1} - u_i^{j+1}}{h} = \frac{u_{i-1}^{j+1} - 2u_i^{j+1} + u_{i+1}^{j+1}}{h^2},$$

where j and τ are the node number and the step along the axis x, and i and h are the node number and the step along the axis y ($i = 2, \dots, n-1$ and $j = 2, \dots, m$) with the boundary conditions $u_2^j - u_1^j = \alpha h$, $u_n^j = 1$, $u_i^1 = u_0(ih)$, and $v_1^j = 0$.

To find the values of v_i^1, we exclude u_x from system (4.1) and obtain the relation $-uv_y + vu_y = u_{yy}$, which should hold up to the line $x = 0$. With allowance for $v_1^1 = 0$, we find v_i^1 from its difference analog

$$u_i^1 \frac{v_{i+1}^1 - v_i^1}{h} + v_i^1 \frac{u_{i+1}^1 - u_i^1}{h} = \frac{u_{i-1}^1 - 2u_i^1 + u_{i+1}^1}{h^2}.$$

A series of calculations, in which α was varied from 0 to 1, was performed. Figure 4 shows the distance l/l_* downstream before the occurrence of a counterflow from the value of α on a logarithmic scale. One can see that the graph is almost a straight line and, hence, one can assume that $\ln(l/l_*) \approx -1.25 - 1.98 \ln \alpha$; we

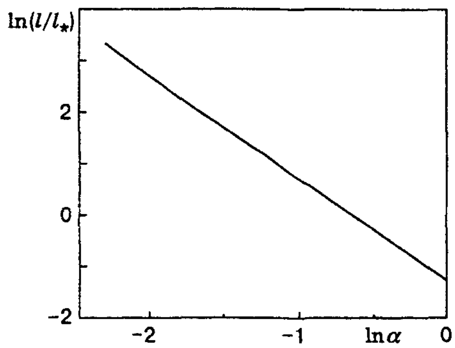

Fig. 4

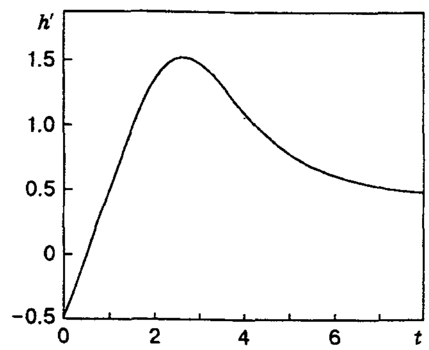

Fig. 5

obtain the approximate formula

$$l \approx \frac{\nu}{U}\frac{0.285}{\alpha^{1.98}} \approx \frac{\nu}{U}\frac{0.285}{\alpha^2}.$$

5. Boundary Layers Conjugate to the Main Layer at Rest. Let $U \equiv 0$ and $f(x) = ax^\gamma$. One can search for the stream function in the form

$$\psi(x,y) = \Big(\frac{a\nu}{\varrho}\Big)^{1/3} x^{(2+\gamma)/3} h(t); \qquad t = \Big(\frac{a}{\varrho\nu^2}\Big)^{1/3} y x^{(\gamma-1)/3}.$$

For $h(t)$, the boundary-value problem

$$\frac{2\gamma+1}{3}h'^2 - \frac{\gamma+2}{3}hh'' = h''', \tag{5.1}$$

$$h(0) = 0, \qquad h''(0) = 1, \qquad h' \to 0 \quad \text{as} \quad t \to \infty \tag{5.2}$$

arises. If $\gamma = -2$, this problem is solved explicitly and the corresponding solution of problem (2.1)–(2.3) has the form

$$u = -\frac{6}{x}\Big(\frac{a^2}{\varrho^2\nu}\Big)^{1/3}\frac{1}{[(a/(\varrho\nu^2))^{1/3}y/x + \sqrt{12}]^3}, \quad v = -\frac{6y}{x^2}\Big(\frac{a^2}{\varrho^2\nu}\Big)^{1/3}\frac{1}{[(a/(\varrho\nu^2))^{1/3}y/x + \sqrt{12}]^3}.$$

Here the perturbation thickness is $\delta = 3\sqrt{12}x(\varrho\nu^2/a)^{1/3}$.

Section 3 dealt with the motions with counterflows which are due to the competition between two physical factors: the pressure gradient and the tangent stress on a free surface. For dynamic reasons, if the boundary layer is conjugate to the main layer at rest, counterflows can occur. For example, let $\gamma = -1$. Then, Eq. (5.1) can be reduced to a Riccati equation of the form $h' = -h^2/6 - t + h'(0)$ by double integration with allowance for the boundary values (5.2). If $h'(0) < 0$, this equation has a solution $h(t)$ such that $h'(t) < 0$ for $t \in (0, \delta)$ and $h'(t) > 0$ for $t > \delta$. In addition, $h = O(1/\sqrt{t})$ as $t \to \infty$.

The example of this solution is shown in Fig. 5, where one can see the graph of the function $h'(t)$. It is noteworthy that the perturbation thickness cannot be calculated in this case: the integral in the right-hand term of the definition (1.3) is divergent.

6. Boundary Layer Near the Three-Phase Contact Point. The motion region can have rigid and free boundaries intersecting at a certain angle. If the fluid motion is quite intense, one can separate boundary layers in the vicinity of the rigid and free boundaries. The problem of the possibility of passing a boundary layer through the contact point has not yet been investigated. Mathematically, this problem is reduced to the problem of separation of an asymptotic (for large Reynolds numbers) form of the Navier–Stokes equations which is applicable to describe the flow near the contact point.

Let the motion occur in the region occupying the quadrant $x > 0$, $y > 0$; the line $\{x = 0\}$ is the rigid wall, and the line $\{y = 0\}$ is the free boundary. We assume that the kinematic and dynamic conditions at these boundaries have the form

$$u\Big|_{x=0} = v\Big|_{x=0} = 0, \qquad \varrho\nu \frac{\partial u}{\partial y}\Big|_{y=0} = f(x), \qquad v\Big|_{y=0} = 0. \tag{6.1}$$

Let ξ and η be an arbitrary system of curvilinear orthogonal coordinates, and v_ξ and v_η be the velocity-vector components in this system. Then the Navier–Stokes equations have the form [5]

$$\frac{v_\xi}{H_\xi}\frac{\partial v_\xi}{\partial \xi} + \frac{v_\eta}{H_\eta}\frac{\partial v_\xi}{\partial \eta} + \frac{v_\eta}{H_\xi H_\eta}\Big(v_\xi \frac{\partial H_\xi}{\partial \eta} - v_\eta \frac{\partial H_\eta}{\partial \xi}\Big) = -\frac{1}{\varrho H_\xi}\frac{\partial p}{\partial \xi} + \nu\Big[\frac{1}{H_\xi^2}\frac{\partial^2 v_\xi}{\partial \xi^2} + \frac{1}{H_\eta^2}\frac{\partial v_\xi}{\partial \eta^2}$$
$$+\frac{1}{H_\xi H_\eta}\frac{\partial(H_\eta/H_\xi)}{\partial \xi}\frac{\partial v_\xi}{\partial \xi} + \frac{1}{H_\xi H_\eta}\frac{\partial (H_\xi/H_\eta)}{\partial \eta}\frac{\partial v_\xi}{\partial \eta} + \frac{2}{H_\xi^2 H_\eta}\frac{\partial H_\xi}{\partial \eta}\frac{\partial v_\eta}{\partial \xi} - \frac{2}{H_\xi H_\eta^2}\frac{\partial H_\eta}{\partial \xi}\frac{\partial v_\eta}{\partial \eta}$$
$$+\frac{1}{H_\xi}\frac{\partial}{\partial \xi}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}\Big)v_\xi + \frac{1}{H_\eta}\frac{\partial}{\partial \eta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta}\Big)v_\xi + \frac{1}{H_\xi}\frac{\partial}{\partial \xi}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta}\Big)v_\eta - \frac{1}{H_\eta}\frac{\partial}{\partial \eta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}\Big)v_\eta\Big]; \tag{6.2}$$

$$\frac{v_\xi}{H_\xi}\frac{\partial v_\eta}{\partial \xi} + \frac{v_\eta}{H_\eta}\frac{\partial v_\eta}{\partial \eta} - \frac{v_\xi}{H_\xi H_\eta}\Big(v_\xi \frac{\partial H_\xi}{\partial \eta} - v_\eta \frac{\partial H_\eta}{\partial \xi}\Big) = -\frac{1}{\varrho H_\eta}\frac{\partial p}{\partial \eta} + \nu\Big[\frac{1}{H_\xi^2}\frac{\partial^2 v_\eta}{\partial \xi^2} + \frac{1}{H_\eta^2}\frac{\partial v_\eta}{\partial \eta^2}$$
$$+\frac{1}{H_\xi H_\eta}\frac{\partial(H_\eta/H_\xi)}{\partial \xi}\frac{\partial v_\eta}{\partial \xi} + \frac{1}{H_\xi H_\eta}\frac{\partial (H_\xi/H_\eta)}{\partial \eta}\frac{\partial v_\eta}{\partial \eta} - \frac{2}{H_\xi^2 H_\eta}\frac{\partial H_\xi}{\partial \eta}\frac{\partial v_\eta}{\partial \xi} + \frac{2}{H_\xi H_\eta^2}\frac{\partial H_\eta}{\partial \xi}\frac{\partial v_\xi}{\partial \eta}$$
$$+\frac{1}{H_\xi}\frac{\partial}{\partial \xi}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}\Big)v_\eta + \frac{1}{H_\eta}\frac{\partial}{\partial \eta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta}\Big)v_\eta - \frac{1}{H_\xi}\frac{\partial}{\partial \xi}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\xi}{\partial \eta}\Big)v_\xi + \frac{1}{H_\eta}\frac{\partial}{\partial \eta}\Big(\frac{1}{H_\xi H_\eta}\frac{\partial H_\eta}{\partial \xi}\Big)v_\xi\Big]; \tag{6.3}$$

$$H_\eta\frac{\partial v_\xi}{\partial \xi} + H_\xi\frac{\partial v_\eta}{\partial \eta} + v_\xi\frac{\partial H_\eta}{\partial \xi} + v_\eta\frac{\partial H_\xi}{\partial \eta} = 0 \tag{6.4}$$

where H_ξ and H_η are the Lamé coefficients of the corresponding coordinates.

We assume that $\xi = (x^2 - y^2)/2$ and $\eta = xy$. Then we have $d\xi = xdx - ydy$ and $d\eta = ydx + xdy$; therefore, $dx^2 + dy^2 = (d\xi^2 + d\eta^2)/(x^2 + y^2)$ or $H_\xi = H_\eta = (x^2+y^2)^{-1/2} = (\xi^2+\eta^2)^{-1/4}/\sqrt{2}$.

We consider the orders of magnitudes in the given problem. Let the stress at the free surface initiate the fluid motion with large Reynolds numbers. Let the order of velocity be denoted by V, l be the characteristic dimension of the region, δ be the thickness of the layer of large velocity gradients at the boundaries of the region of motion at a distance from them (the boundary layer), and F be the order of $f(x)$. Similarly to [3], it is easy to estimate the values of V, Re, and δ. From the equations of motion and the boundary conditions, we obtain the ordinal relations $V^2/l = \nu V/\delta^2$ and $\varrho\nu V/\delta = F$; as a result, we have

$$\delta = \Big(\frac{l\varrho\nu^2}{F}\Big)^{1/3}, \qquad V = \frac{F\delta}{\varrho\nu}, \qquad \mathrm{Re} = \Big(\frac{l^4F^2}{\varrho^2\nu^4}\Big)^{1/3}.$$

Therefore, if Re is large, we have $\delta = l/\sqrt{\mathrm{Re}} \ll l$, and $\delta/l \ll 1$.

We separate the asymptotic form of Eqs. (6.2)–(6.4) under the assumption that $v_\xi \sim V$, $v_\eta \sim \delta V/l$, $\xi \sim l^2$, and $\eta \sim \delta l$. The above example, which is important for applications, shows that these orders of magnitude are quite real. Here $H_\xi \sim H_\eta \sim 1/l$. Keeping the higher-order terms relative to the δ/l, we obtain an analog of the system of Prandtl equations:

$$v_\xi\frac{\partial v_\xi}{\partial \xi} + v_\eta\frac{\partial v_\xi}{\partial \eta} = -\frac{1}{\varrho}\frac{\partial p}{\partial \xi} + \nu\sqrt{2|\xi|}\,\frac{\partial^2 v_\xi}{\partial \eta^2}, \tag{6.5}$$

$$\frac{\partial p}{\partial \eta} = 0, \qquad \frac{\partial v_\xi}{\partial \xi} + \frac{\partial v_\eta}{\partial \eta} - \frac{v_\xi}{2\xi} = 0. \tag{6.6}$$

As $\mathrm{Re} \to \infty$, the boundary conditions (6.1) take the form

$$v_\xi, v_\eta\Big|_{\eta=0,\xi<0} = 0, \qquad \frac{\varrho\nu}{2\sqrt{|\xi|}}\frac{\partial v_\xi}{\partial \eta}\Big|_{\eta=0,\xi>0} = f(\xi), \qquad v_\eta\Big|_{\eta=0,\xi>0} = 0. \tag{6.7}$$

By virtue of the first equation in (6.6), one can assume that $p = p(\xi)$ and introduce $U(\xi)$, the velocity of the external (potential) motion at the frontiers of the region. Here the case $\xi < 0$ corresponds to the boundary $\{x = 0\}$, and $\xi > 0$ to the boundary $\{y = 0\}$.

Assuming that $f \leqslant 0$, we obtain that problem (6.5)–(6.7) corresponds to the case of the transition of the Prandtl boundary layer to the Marangoni layer. Obviously, alongside with conditions (6.7), we can consider the conditions

$$\frac{\varrho\nu}{2\sqrt{|\xi|}}\frac{\partial v_\xi}{\partial\eta}\Big|_{\eta=0,\xi<0} = f(\xi), \qquad v_\eta\Big|_{\eta=0,\xi<0} = 0, \qquad v_\xi, v_\eta\Big|_{\eta=0,\xi>0} = 0, \tag{6.7'}$$

which correspond to the case where the Marangoni becomes a Prandtl layer. Here the fluid runs into a rigid wall.

It is noteworthy that, for $\xi = 0$, there should be $U = 0$. In addition, the pressure gradient $\partial p/\partial\xi$ can have a discontinuity of the first kind for $\xi = 0$. In this case, the solution of problem (6.5)–(6.7) is understood in the sense of [8], where the Prandtl boundary layer with a discontinuous pressure gradient is considered.

The second equation in (6.6) allows one to introduce a stream function ψ such that $v_\xi = \sqrt{|\xi|}\,\partial\psi/\partial\eta$ and $v_\eta = -\sqrt{|\xi|}\,\partial\psi/\partial\xi$. Passing to the Mises variables ξ, ψ, and $w = v_\xi^2$, we obtain the equation

$$\frac{\partial w}{\partial\xi} = -\frac{2}{\varrho}\frac{dp}{d\xi} + \frac{\nu\sqrt{2w}}{\sqrt{|\xi|}}\frac{\partial^2 w}{\partial\psi^2},$$

which is reduced, by the replacement $s = \int_{\xi_0}^{\xi}\sqrt{2/|\xi|}\,d\xi$, to the classical Mises equation of boundary-layer theory

$$\frac{\partial w}{\partial s} = -\frac{2}{\varrho}\frac{dp}{ds} + \nu\sqrt{w}\frac{\partial^2 w}{\partial\psi^2}.$$

Here the boundary conditions (6.7) take the form

$$w\Big|_{\psi=0,s<s_0} = 0, \qquad \varrho\nu\frac{\partial w}{\partial\psi}\Big|_{\psi=0,s>s_0} = f(s).$$

The conditions under which the Marangoni layer becomes a Prandtl layer can be written similarly.

We shall consider some examples. If one sets $f(\xi) = -F\xi^{3\beta+1}$ and $U^2(\xi) = 2P|\xi|^{4\beta+3}/[\varrho(4\beta+3)]$ in problem (6.5)–(6.7), its solution can be searched for in the form

$$\psi = \left(\frac{2P\nu^2}{\varrho}\right)^{1/4}|\xi|^{\beta+1}h(t), \qquad t = \left(\frac{P}{2\varrho\nu^2}\right)^{1/4}|\xi|^{\beta}\eta. \tag{6.8}$$

Since there should be $U(\xi) = 0$, we have $\beta > -3/4$. Substituting this representation into problem (6.5)–(6.7), we obtain that its solution has the form (6.8) for $\xi < 0$, where $h(t)$ is the solution of the problem

$$h''' = 1 - \frac{4\beta+3}{2}h'^2 + (\beta+1)hh''; \tag{6.9}$$

$$h(0) = h'(0) = 0, \qquad h'(t) \to \sqrt{\frac{2}{4\beta+3}} \quad \text{for} \quad t \to \infty. \tag{6.10}$$

For $\xi = 0$, we have $v_\xi = v_\eta = 0$, and, for $\xi > 0$, the solution again has the form (6.8), where $h(t)$ is the solution of the problem

$$h''' = -1 + \frac{4\beta+3}{2}h'^2 - (\beta+1)hh''; \tag{6.11}$$

$$h(0) = 0, \qquad h''(0) = -\mu, \qquad h'(t) \to \sqrt{\frac{2}{4\beta+3}} \quad \text{as} \quad t \to \infty. \tag{6.12}$$

If $\beta > -1/2$, the resulting solution of problem (6.5)–(6.7) is smooth everywhere. If $\beta = -1/2$, the pressure gradient $dp/d\xi$ for $\xi = 0$ has a discontinuity of the first kind and the solution should be understood in the sense of [8]. In this case, the solution is not smooth only in the coordinates ξ and η and smooth in the Cartesian coordinates. If $-4/3 < \beta < -1/2$, for $\xi = 0$, the pressure gradient becomes infinite. Special studies are needed to understand whether these solutions have meaning. If the thermocapillary effect is the source of motion (as during crucibleless zone melting at zero gravity), one can expect that the case $\beta = -1/2$ is most frequent. Problems (6.9), (6.10) and (6.11), (6.12) were solved numerically for $\beta = -1/2$ and $\beta > -1/2$. The behavior of the velocity distribution $\dot{h}(t)$ has a form conventional for boundary layers [2, 5]. The examples show that the boundary problem (6.5)–(6.7) can be used for analysis of the flows near the three-phase contact point.

7. Conclusions. In studying the occurrence of a counterflow, it seems natural to find a relation between the parameters of a problem in which the free boundary is fixed. Then, changing this relation, one can expect the presence of counterflows and standard solutions with a unidirectional longitudinal velocity.

In the present study, the cases where the main flow has a constant velocity or is characterized by a constant pressure gradient (the case of a varied pressure gradient differs little from the latter case) were considered in detail.

In the case $U =$ const, the equilibrium on the free surface was reached for the tangent stress $f(x) = F/\sqrt{x}$ if $U = U_*$ (the constant U_* was determined in Sec. 1). For each $U > U_*$, two solutions, one of which cannot take place physically and the other has no counterflow, were constructed. Additional studies of the Cauchy problem for the Blasius equation (they are of independent value) showed that, for $U < U_*$, there are no self-similar solutions of the problem.

Another situation arose when the longitudinal pressure gradient in the main flow was equal to the constant $-P$. A value of $P = P_0$ is also calculated at which equilibrium was observed at the free boundary, but solutions having physical meaning exist for both $P > P_0$ and $P < P_0$. In the latter case, the flow region contained a counterflow zone involving a larger part of the boundary layer the smaller the value of P.

The case of the constant velocity of the external flow and the tangent stress is characterized by the fact that the return-flow region always occurs downstream with distance from the coordinate origin, because the equilibrium of the free boundary requires a decrease in the stress downstream in inverse proportion to the square root of the distance. An approximate formula to calculate the distance to the counterflow zone was derived.

The motion region can have rigid walls intersecting at a certain angle and free boundaries. In solving the problem by the method of selecting the boundary layers, the question arises whether it is possible to pass the boundary layer through the contact point. The asymptotic (for large Reynolds numbers) form of the Navier–Stokes equations which is applicable to the description of flows near the contact point was found. Examples of the calculation were given.

The author thanks G. B. Volkova for her assistance in carrying out the calculations.

This work was supported by the Russian Foundation for Fundamental Research (Grant No. 97-01-00818).

REFERENCES

1. L. G. Napolitano, "Marangoni boundary layers," in: *Proc. 3rd Europ. Symp. on Material Sci. in Space*, Grenoble (1979), 349–358.
2. L. G. Napolitano and G. Golia, "Coupled Marangoni boundary layers," *Acta Astronaut.*, **8**, Nos. 5/6, 417–434 (1981).
3. V. A. Batishchev, V. V. Kuznetsov, and V. V. Pukhnachov, "Marangoni boundary layers," *Prog. Aerospace Sci.*, **26**, 353–370 (1989).
4. V. V. Kuznetsov, "The existence of a boundary layer near a free surface," in: *Dynamics of Continuous Media* (collected scientific papers) [in Russian], Novosibirsk, **67**, (1984), pp. 68–75.

5. N. E. Kochin, I. A. Kibel', and N. V. Rose, *Theoretical Hydromechanics* [in Russian], Part 2, Gostekhteorizdat, Moscow (1948).
6. F. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York–London–Sydney (1964).
7. C. A. J. Fletcher, *Computational Techniques for Fluid Dynamics* [Russian translation], Vol. 1, Mir, Moscow (1991).
8. T. D. Dzhuraev, "The system of equations of boundary-layer theory for a steady-state flow of an incompressible fluid," *Differ. Uravn.*, **4**, No. 11, 2068–2083 (1968).